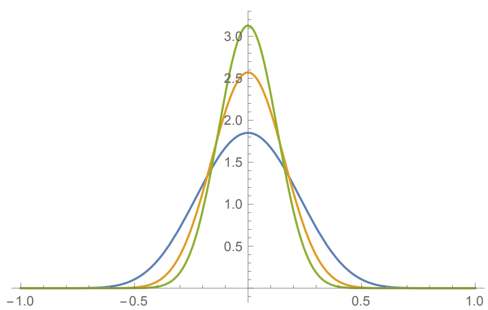
Sequences and Series of Functions (Rudin)

Stone-Weierstrass Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists a sequence of polynomials $(P_n(x))$ such that converges uniformly to f on $[a, b]$.

Proof: Consider any continuous function $g : \mathbb{R} \to \mathbb{R}$ that satisfies $g(x) = 0$ for $x \notin [0, 1]$. For each $n \geq 1$ set

$$Q_n(x) = \frac{(1 - x^2)^n}{\int_{-1}^{1} (1 - x^2)^n \, dx}.$$

The area under the curve $y = Q_n(x)$ over $[-1, 1]$ is equal to 1, and has the shape of a bell curve with most of it's area concentrated in a narrow band above $x = 0$. The polynomials $Q_{10}(x)$, $Q_{20}(x)$, $Q_{30}(x)$ are plotted below:



For $0 \leq x \leq 1$ define

$$P_n(x) = \int_{-1}^{1} g(x + t) Q_n(t) \, dt.$$

$P_n(x)$ is a polynomial of degree $\leq 2n$:

$$\int_{-1}^{1} g(x + t) t^k \, dt = \int_{x-1}^{x+1} g(u)(u - x)^k \, du = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^i \int_{x-1}^{x+1} g(u) u^{k-i} \, du.$$

We have

$$|P_n(x) - g(x)| \leq \int_{-1}^{1} |g(x + t) - g(x)| Q_n(t) \, dt.$$

Let $M = \sup g(x)$. Since g is uniformly continuous on $\mathbb{R}$, there exists $\delta_k > 0$ such that $|t| < \delta_k$ implies $|g(x+t) - g(x)| \leq \frac{1}{k}$ for all x. This yields

$$|P_n(x) - g(x)| \leq 2M \int_{-1}^{-\delta_k} Q_n(t)\ dt + \frac{1}{k} \int_{-\delta_k}^{\delta_k} Q_n(t)\ dt + 2M \int_{\delta_k}^{1} Q_n(t)\ dt.$$

We have

$$\frac{1}{k} \int_{-\delta_k}^{\delta_k} Q_n(t)\ dt \leq \frac{1}{k}.$$

We also have

$$\int_{-1}^{1} (1-x^2)^n\ dx \geq 2 \int_{0}^{1} (1-x)^n\ dx = \frac{2}{n+1},$$

hence for $\delta_k \leq |x| \leq 1$ we have

$$|Q_n(x)| \leq \frac{n+1}{2}(1-\delta_k^2)^n.$$

Therefore

$$|Pn(x) - g(x)| \leq 2M(n+1)(1-\delta_k^2)^n + \frac{1}{k}$$

for $0 \leq x \leq 1$, i.e.

$$||P_n - g|| \leq 2M(n+1)(1-\delta_k^2)^n + \frac{1}{k}.$$

Let $\epsilon > 0$ be given. Choose k so that $\frac{1}{k} < \frac{\epsilon}{2}$. For sufficiently large n, $2M(n+1)(1-\delta_k^2)^n < \frac{\epsilon}{2}$, hence $||P_n - g|| < \epsilon$. Therefore $P_n \to g$ uniformly on $[0, 1]$.

Given $f : [a, b] \to \mathbb{R}$, $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$ satisfies $g(a) = g(b) = 0$ and $h(x) = g((b-a)x + a)$ satisfies $h(0) = h(1) = 0$. If $P_n(x) \to h(x)$ uniformly on $[0, 1]$, then $P_n(x) \to g((b-a)x + a)$ on $[0, 1]$, therefore $P_n(\frac{x-a}{b-a}) \to g(x)$ uniformly on $[a, b]$, therefore

$$P_n(\frac{x-a}{b-a}) + f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \to f(x)$$

uniformly on $[a, b]$.

Corollary: If $f(0) = 0$ and $P_n(x) \to f$ uniformly on $[-a, a]$, then $P_n(0) \to 0$, hence $P_n(x) - P_n(0) \to f$ uniformly on $[-a, a]$. So f can be uniformly approximated by a polynomial with zero constant term.

Defintion: An algebra $\mathbb{A}$ of functions $f : E \to \mathbb{R}$ is a set of functions closed under addition, multiplication, and scalar multiplication. An algebra $\mathbb{A}$ is said to separate points on E if for each $x \neq y$ in E there exist $f, g \in \mathbb{A}$ such that $f(x) \neq g(y)$, and to vanish at $x \in E$ if $f(x) = 0$ for all $f \in \mathbb{A}$.

Theorem: Let $\mathbb{A}$ be an algebra of bounded functions from E to $\mathbb{R}$. Let $||f|| = \sup_{x \in E} f(x)$. Then $d(f, g) = ||f - g||$ is a metric on $\mathbb{A}$ and $\overline{\mathbb{A}}$ is an algebra of bounded functions from E to $\mathbb{R}$.

Proof: Clearly $||f - f|| = 0$, $f \neq g \implies ||f - g|| > 0$, $||f - g|| = ||g - f||$. Now let $f, g, h \in \mathbb{A}$ be given. For any $x \in E$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f| + ||g||,$$

therefore $||f + g|| \leq ||f|| + ||g||$. This implies

$$||f - h|| = ||(f - g) + (g - h)|| \leq ||f - g|| + ||g - h||.$$

Also, for any $x \in E$,

$$|f(x)g(x)| \leq ||f|||g(x)| \leq ||f||||g||,$$

therefore $||fg|| \leq ||f||||g||$. Clearly $||cf|| = |c|||f||$ for all $c \in \mathbb{R}$.

Now suppose $f_n \to f$, $g_n \to g$, and $c \in \mathbb{R}$. Then

$$||f_n + g_n - f - g|| = ||f_n - f|| + ||g_n - g|| \to 0,$$

hence $f_n + g_n \to f + g$. Also,

$$||f_n g_n - fg|| \leq ||f_n g_n - f_n g|| + ||f_n g - fg|| \leq ||f_n||||g_n - g|| + ||f_n - f||||g|| \to 0,$$

therefore $f_n g_n \to fg$. Finally,

$$||cf_n - cf|| = |c|||f_n - f|| \to 0,$$

therefore $cf_n \to cf$.

Given $f_n \to f$, for any $x \in E$ we have

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq ||f - f_n|| + ||f_n|| \leq 1 + ||f_n||$$

for $n \geq n_0$. Therefore $||f|| \leq 1 + ||f_{n_0}||$.

Lemma: Let $u \neq v \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$ be given. Then there exists $p(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ such that $p(u) = a$ and $p(v) = b$.

Proof: Set $q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 + 1$. Since $u \neq v$, $x_k(u) \neq x_k(v)$ for some k. We can set

$$p(x_1, \ldots, x_n) = \frac{aq(x_1, \ldots, x_n)(x_k - v_k)}{q(u_1, \ldots, u_n)(u_k - v_k)} + \frac{bq(x_1, \ldots, x_n)(x_k - u_k)}{q(u_1, \ldots, u_n)(v_k - u_k)}$$

Theorem: Let $K \subseteq \mathbb{R}^n$ be a compact set. Then a function $f : K \to \mathbb{R}$ is continuous if and only if there exists a sequence (f_n) in $\mathbb{R}[x_1, \ldots, x_n]$ such that $f_n \to f$ uniformly on K.

Proof: Let $\mathbb{A}$ be the set of polynomial functions on K. Suppose $f_n \to f$ uniformly, where each $f_m \in \mathbb{A}$. Then f is continuous: Let $\epsilon > 0$ be given. Choose n so that $||f_n - f|| < \frac{\epsilon}{3}$. Since f_n is continuous on K and K is compact, f_n is uniformly continuous on K, hence there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Hence $|x - y| < \delta$ implies

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq$$

$$||f - f_n|| + \frac{\epsilon}{3} + ||f_n - f|| < \epsilon.$$

Conversely, let $f : K \to \mathbb{R}$ be continuous. We will show that $f \in \overline{\mathbb{A}}$ as follows:

1. For all $g \in \overline{\mathbb{A}}$, $|g| \in \overline{\mathbb{A}}$. Proof: Let $g \in \overline{\mathbb{A}}$ and $\epsilon > 0$ be given. Choose a polynomial $P(x)$ with zero constant term such that $||P(x) - | \cdot ||| < \frac{\epsilon}{2}$ on $[-||g||, ||g||]$. Then $||P(g) - |g||| < \frac{\epsilon}{2}$. Given $g_n \to g$, we have $P(g_n) \to P(g)$. Choose n so that $||P(g_n) - P(g)|| < \frac{\epsilon}{2}$. Hence $||P(g_n) - |g||| < \epsilon$. Since $P(g_n) \in \mathbb{A}$, $|g| \in \overline{\mathbb{A}}$.

2. If $g, h \in \mathbb{A}$, then $\max(g, h) \in \overline{\mathbb{A}}$ and $\min(g, h) \in \overline{\mathbb{A}}$. Proof:

$$\max(g, h) = \frac{g + h}{2} + \frac{|g - h|}{2},$$

$$\min(g, h) = g + h - \max(g, h).$$

3. Fix $x \in K$. Then there exists $f_x \in \overline{\mathbb{A}}$ such that $f_x(x) = f(x)$ and

$$f_x(k) > f(k) - \epsilon$$

for all $k \in K$. Proof: For each $y \in K$ choose $g_y \in \mathbb{A}$ such that $g_y(x) = f(x)$ and $g_y(y) = f(y)$. Then $y \in (g_y - f)^{-1}(\epsilon, \infty)$, hence

$$K = \bigcup_{y \in K} \{(g_y - f)^{-1}((-\epsilon, \infty)) : y \in K\},$$

hence by compactness of K there exist $y_1, \ldots, y_a \in K$ such that for each $k \in K$ there exists i such that $g_{y_i}(k) > f(k) - \epsilon$. We can set $f_x = \max(g_{y_1}, \ldots, g_{y_a})$. In particular, $f_x(x) = x$.

4. We $x \in (f_x - f)^{-1}((-\infty, \epsilon))$ for each $x \in K$, hence

$$K = \bigcup_{x \in K} \{(f_x - f)^{-1}((-\infty, \epsilon)) : y \in K\},$$

hence by compactness of K there exist $x_1, \ldots, x_b \in K$ such that for each $k \in K$ there exists i such that $f_{x_i}(k) < f(k) + \epsilon$. Setting $f_\epsilon = \min(f_{x_1}, \cdots, f_{x_b}) \in \overline{\mathbb{A}}$, we have $||f_\epsilon - f|| < \epsilon$. Since ϵ is arbitrary, $f \in \overline{\mathbb{A}}$.